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EVALUATION OF AN ENERGY METHOD USING FINITE DIFFERENCES FOR DETERMINING THERMAL MIDPLANE STRESSES IN PLATES

by Harry G. Schaeffer and Walter L. Heard, Jr.

Langley Research Center

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SUMMARY

A method of obtaining a set of finite-difference expressions for the thermal stress function directly from the complementary strain-energy expression is evaluated. The essential difference between this method and the more usual partial-differential-equation approach is that, in the energy approach, only conditions involving the variation of the stress function and its first normal derivative at the boundary must be satisfied explicitly. These conditions are generally termed the "geometric" boundary conditions. In approaching the problem from the partial-differential-equation viewpoint, rather than from the energy viewpoint, it is necessary to satisfy explicitly additional conditions involving the second and third derivatives which are generally termed "natural" boundary conditions.

The finite-difference method is evaluated by applying it to a thermal stress problem for which an approximate solution and experimental results are available. The finite-difference results appear to converge rapidly as the number of grid stations is increased. A collection of coefficients for grid stations on or near the boundary is presented to aid in the rapid formulation of specific problems.

INTRODUCTION

In many applications of current interest - for example, the structural skin of vehicles designed for high-speed flight in the atmosphere - platelike configurations are subjected to a thermal environment. This environment, together with restraints on the plate boundary, may induce stress in the midplane of the plate.

The problem of determining the relationship between midplane stress distribution and temperature distribution has received considerable attention in the past. In reference 1 a variational method is used to determine the approximate relationship between temperature distribution and midplane stress for thin, flat, rectangular plates. In reference 2 a similar problem is solved by using the characteristic functions for beam vibration modes. A finite-difference method is presented in reference 3 which is based on an extended stiffness method.

In references 4 and 5 finite-difference methods are presented in which the necessary set of finite-difference equations is obtained directly from the applicable energy expression. For certain problems this technique presents a much simpler approach for obtaining the desired set of difference equations than does the more usual finite-difference representation of the partial differential equation. The particular technique discussed in reference 5 is evaluated in reference 6 for plate vibration problems. The purpose of the present paper is to evaluate this technique for determining the thermal-stress-equilibrium state of a plate where the forcing function consists of a temperature distribution.

SYMBOLS

a	one-half of plate length (see fig. 3)
b	one-half of plate width (see fig. 3)
E	modulus of elasticity, function of x and y
E_r	reference modulus of elasticity
$\{G\}$	thermal-forcing-function column matrix
G_i	element of $\{G\}$
h	plate thickness, function of x and y
h_r	reference plate thickness
M	total number of half-stations associated with type 2 areas
K	total number of grid stations associated with type 1 areas
N_x	x -direction midplane force per unit length
N_y	y -direction midplane force per unit length
N_{xy}	midplane shear force per unit length
$[Q]$	general matrix of coefficients
Q_{ij}	element of $[Q]$
R	number of grid stations at which solution is unknown
$T_i = E_r h_r (\alpha \Delta T \eta)_i \epsilon^2$	$(i = 1, 2, \dots, K)$

ΔT temperature change from reference-temperature state, function of x and y

U complementary strain energy

x, y Cartesian coordinates of plate

α coefficient of thermal expansion, function of x and y

$$\beta_i = E_R h_R \left(\frac{\eta}{Eh} \right)_i \quad (i = 1, 2, \dots, K)$$

$$\bar{\beta}_j = E_R h_R \left(\frac{\eta}{Eh} \right)_j \quad (j = 1, 2, \dots, M)$$

ϵ grid spacing in x -direction

$\eta_1, \eta_j, \eta_m, \eta_k$ integration factors

λ grid spacing in y -direction

μ Poisson's ratio

ϕ Airy stress function

$[\phi]$ column matrix of ϕ values to be determined

$\delta()$ variational operator

∇^2 Laplacian operator

Subscripts:

$N, W, NN, EE, \dots, NE/2, NW/2, \dots$ grid points, located with respect to a general station (see fig. 2)

0 indicates evaluation of a quantity at a general grid point (see fig. 2)

i, j, m, k integers

GENERAL METHOD OF ANALYSIS

The usual finite-difference approach is to replace the partial derivatives in the governing partial differential equation with a suitable set of finite-difference approximations. For points on or near the boundary, the derivatives

are expressed in terms of fictitious exterior points; these fictitious points are then eliminated by means of the boundary conditions, also expressed in finite-difference form.

The essential difference between the usual finite-difference approach based on the partial differential equation and the energy approach lies in the manner of satisfying the boundary conditions to be applied to the solution surface, the surface in space defined by the Airy stress function $\phi = \phi(x, y)$. In the energy approach the only conditions that must be satisfied explicitly are those involving the variation of the solution surface and the variation of certain first normal derivatives of the solution surface at the boundary. These conditions are generally termed the "geometric," or forced, boundary conditions. The remaining boundary conditions, generally termed the "natural" boundary conditions, are formulated in terms of the second and third derivatives of the solution surface. These conditions, which are often difficult to handle explicitly in the usual finite-difference approach, are automatically satisfied as accurately as possible by using the energy approach.

DERIVATION OF DIFFERENCE EQUATIONS

For a heated plate the appropriate energy expression obtained from reference 5 is

$$U = \iint \frac{1}{2Eh} \left[\left(\frac{\partial^2 \phi}{\partial x^2} \right)^2 + \left(\frac{\partial^2 \phi}{\partial y^2} \right)^2 - 2\mu \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} + 2(1 + \mu) \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right. \\ \left. + 2\alpha E h \Delta T \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \right] dx \, dy \quad (1)$$

where ϕ is the Airy stress function which has been introduced to satisfy the equilibrium conditions and is related to the stress resultants by the following equations:

$$\left. \begin{aligned} N_y &= \frac{\partial^2 \phi}{\partial x^2} \\ N_x &= \frac{\partial^2 \phi}{\partial y^2} \\ N_{xy} &= - \frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \right\} \quad (2)$$

The application of the calculus of variations to equation (1) leads to the following differential equation for a rectangular plate:

$$\nabla^4 \phi + \alpha E h \nabla^2 (\Delta T) = 0 \quad (3)$$

and to the following associated boundary conditions:

Along $x = x_1, x = x_2$:

$$\frac{\partial^2 \phi}{\partial x^2} - \mu \frac{\partial^2 \phi}{\partial y^2} + \alpha E h \Delta T = 0 \quad \text{or} \quad \delta \left(\frac{\partial \phi}{\partial x} \right) = 0, \quad (N_{xy} = 0)$$

$$\frac{\partial^3 \phi}{\partial x^3} + (2 + \mu) \frac{\partial^3 \phi}{\partial x \partial y^2} + \alpha E h \frac{\partial (\Delta T)}{\partial x} = 0 \quad \text{or} \quad \delta(\phi) = 0, \quad (N_x = 0)$$

Along $y = y_1, y = y_2$:

$$\frac{\partial^2 \phi}{\partial y^2} - \mu \frac{\partial^2 \phi}{\partial x^2} + \alpha E h \Delta T = 0 \quad \text{or} \quad \delta \left(\frac{\partial \phi}{\partial y} \right) = 0, \quad (N_{xy} = 0)$$

$$\frac{\partial^3 \phi}{\partial y^3} + (2 + \mu) \frac{\partial^3 \phi}{\partial x^2 \partial y} + \alpha E h \frac{\partial (\Delta T)}{\partial y} = 0 \quad \text{or} \quad \delta(\phi) = 0, \quad (N_y = 0)$$

At $(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_2, y_2)$:

$$\frac{2(1 + \mu)}{E} \frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad \text{or} \quad \delta(\phi) = 0$$

where $x = x_1, x = x_2, y = y_1$, and $y = y_2$ denote the edges of the plate.

The boundary conditions given in the parentheses are the physical interpretations of the boundary conditions on ϕ and its first derivative and are seen to be force boundary conditions. The boundary conditions involving the second and third derivatives of ϕ are conditions of compatibility which are discussed in detail in reference 5. These boundary conditions do not have to be considered when applying the energy method to obtain directly the finite-difference equations.

The first step in determining the necessary difference equations is to divide the surface of the plate into a suitable grid network as is shown in figure 1 for a plate of arbitrary shape. The intersections of the grid network are referred to as grid stations or grid points. The derivatives in equation (1) are expressed by means of finite-difference expressions in terms of the value of the solution surface associated with each grid station ϕ_i . The

required set of finite-difference equations is then determined by minimizing the complementary strain energy with respect to each ϕ_i .

The notation relating to the grid stations is shown in figure 2. The station 0 is a central point at which the associated unknown value of the solution surface is referred to as ϕ_0 . The subscripts associated with the other grid stations are the directional bearings of the stations with respect to the station 0. Subscripts are also indicated at half-stations which do not coincide with the grid stations, but which are useful in evaluating the mixed partial derivative.

For integration over the plate, elements shown as type 1 area in figure 2 are used for all $\partial^2\phi/\partial x^2$ and $\partial^2\phi/\partial y^2$ terms. Elements designated as type 2 area are used for the $\partial^2\phi/\partial x \partial y$ terms. The following set of finite-difference expressions having the orders of error indicated was used (see chs. II and V of ref. 7):

$$\left. \begin{aligned} \frac{\partial^2\phi_0}{\partial x^2} &= \frac{1}{\epsilon^2}(\phi_W - 2\phi_0 + \phi_E) & o(\epsilon^2) \\ \frac{\partial^2\phi_0}{\partial y^2} &= \frac{1}{\lambda^2}(\phi_N - 2\phi_0 + \phi_S) & o(\lambda^2) \\ \frac{\partial^2\phi_{NE/2}}{\partial x \partial y} &= \frac{1}{\epsilon\lambda}(\phi_{NE} - \phi_N - \phi_E + \phi_0) & o(\epsilon\lambda) \end{aligned} \right\} \quad (4)$$

The replacement of the integration in equation (1) by a trapezoidal summation and subsequent differentiation with respect to each ϕ_i leads to the following general equation:

$$\begin{aligned} \frac{\partial U}{\partial \phi_i} &= \sum_{k=1}^K \epsilon\lambda \left(\frac{\eta}{Eh} \right)_k \left[\left(\frac{\partial^2\phi_k}{\partial x^2} - \mu \frac{\partial^2\phi_k}{\partial y^2} \right) \frac{\partial}{\partial \phi_i} \left(\frac{\partial^2\phi_k}{\partial x^2} \right) + \left(\frac{\partial^2\phi_k}{\partial y^2} - \mu \frac{\partial^2\phi_k}{\partial x^2} \right) \frac{\partial}{\partial \phi_i} \left(\frac{\partial^2\phi_k}{\partial y^2} \right) \right. \\ &\quad \left. + (\alpha E h \eta \Delta T)_k \frac{\partial}{\partial \phi_i} \left(\frac{\partial^2\phi_k}{\partial x^2} + \frac{\partial^2\phi_k}{\partial y^2} \right) \right] + \sum_{m=1}^M 2\epsilon\lambda(1 + \mu) \left(\frac{\eta}{Eh} \right)_m \frac{\partial^2\phi_m}{\partial x \partial y} \frac{\partial}{\partial \phi_i} \left(\frac{\partial^2\phi_m}{\partial x \partial y} \right) = 0 \\ &\quad (i = 1, 2, 3, \dots, R) \quad (5) \end{aligned}$$

where

- R number of grid stations at which value of solution surface is unknown;
 in general, includes fictitious grid points located just off
 boundaries
- K total number of grid stations associated with type 1 areas (fig. 2)
- M total number of grid stations associated with type 2 areas (fig. 2)
- k grid stations for which type 1 areas apply
- m half-stations for which type 2 areas apply

The integration factors, represented by η_k and η_m , are unity for a general point but may be less than unity for points in the neighborhood of a boundary, depending on the degree that the boundary has truncated the grid element.

The required set of difference equations can now be determined by utilizing equations (4) in equation (5). It is noted from equations (4) and (5) that the summation process need not proceed over the entire plate; that is, the partial derivative of the energy with respect to ϕ_i exists only in the neighborhood of the general point i. For example, consider a particular point 0, as shown in figure 2; then the term

$$\frac{\partial}{\partial \phi_0} \left(\frac{\partial^2 \phi_n}{\partial x^2} \right)$$

has nonzero values only when the second derivative is evaluated at 0, E, or W. Therefore, by using equations (4), equation (5) becomes:

$$\begin{aligned}
\frac{\partial U}{\partial \phi_0} = & \beta_0 \left\{ (\phi_E - 2\phi_0 + \phi_W) \frac{\partial}{\partial \phi_0} (\phi_E - 2\phi_0 + \phi_W) + \left(\frac{\epsilon}{\lambda}\right)^4 (\phi_N - 2\phi_0 + \phi_S) \frac{\partial}{\partial \phi_0} (\phi_N - 2\phi_0 + \phi_S) - \mu \left(\frac{\epsilon}{\lambda}\right)^2 \left[(\phi_E - 2\phi_0 + \phi_W) \frac{\partial}{\partial \phi_0} (\phi_N - 2\phi_0 + \phi_S) \right. \right. \\
& \left. \left. + (\phi_N - 2\phi_0 + \phi_S) \frac{\partial}{\partial \phi_0} (\phi_E - 2\phi_0 + \phi_W) \right] \right\} + \beta_N \left[\left(\frac{\epsilon}{\lambda}\right)^4 (\phi_{NN} - 2\phi_N + \phi_0) \frac{\partial}{\partial \phi_0} (\phi_{NN} - 2\phi_N + \phi_0) \right. \\
& - \mu \left(\frac{\epsilon}{\lambda}\right)^2 (\phi_{NE} - 2\phi_N + \phi_{NW}) \frac{\partial}{\partial \phi_0} (\phi_{NN} - 2\phi_N + \phi_0) \left. \right] + \beta_S \left[\left(\frac{\epsilon}{\lambda}\right)^4 (\phi_{SS} - 2\phi_S + \phi_0) \frac{\partial}{\partial \phi_0} (\phi_{SS} - 2\phi_S + \phi_0) \right. \\
& - \mu \left(\frac{\epsilon}{\lambda}\right)^2 (\phi_{SE} - 2\phi_S + \phi_{SW}) \frac{\partial}{\partial \phi_0} (\phi_{SS} - 2\phi_S + \phi_0) \left. \right] + \beta_W \left[(\phi_{WW} - 2\phi_W + \phi_0) \frac{\partial}{\partial \phi_0} (\phi_{WW} - 2\phi_W + \phi_0) \right. \\
& - \mu \left(\frac{\epsilon}{\lambda}\right)^2 (\phi_{NW} - 2\phi_W + \phi_{SW}) \frac{\partial}{\partial \phi_0} (\phi_{WW} - 2\phi_W + \phi_0) \left. \right] + \beta_E \left[(\phi_{EE} - 2\phi_E + \phi_0) \frac{\partial}{\partial \phi_0} (\phi_{EE} - 2\phi_E + \phi_0) \right. \\
& - \mu \left(\frac{\epsilon}{\lambda}\right)^2 (\phi_{NE} - 2\phi_E + \phi_{SE}) \frac{\partial}{\partial \phi_0} (\phi_{EE} - 2\phi_E + \phi_0) \left. \right] + 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 \left[\bar{\beta}_{NE/2} (\phi_{NE} - \phi_N - \phi_E + \phi_0) \frac{\partial}{\partial \phi_0} (\phi_{NE} - \phi_N - \phi_E + \phi_0) \right. \\
& + \bar{\beta}_{SW/2} (\phi_0 - \phi_W - \phi_S + \phi_{SW}) \frac{\partial}{\partial \phi_0} (\phi_0 - \phi_W - \phi_S + \phi_{SW}) + \bar{\beta}_{NW/2} (\phi_N - \phi_{NW} - \phi_0 + \phi_W) \frac{\partial}{\partial \phi_0} (\phi_N - \phi_{NW} - \phi_0 + \phi_W) \\
& + \bar{\beta}_{SE/2} (\phi_E - \phi_0 - \phi_{SE} + \phi_S) \frac{\partial}{\partial \phi_0} (\phi_E - \phi_0 - \phi_{SE} + \phi_S) \left. \right] + T_0 \left[\frac{\partial}{\partial \phi_0} (\phi_E - 2\phi_0 + \phi_W) + \left(\frac{\epsilon}{\lambda}\right)^2 \frac{\partial}{\partial \phi_0} (\phi_N - 2\phi_0 + \phi_S) \right] \\
& + \left(\frac{\epsilon}{\lambda}\right)^2 \left[T_N \frac{\partial}{\partial \phi_0} (\phi_{NN} - 2\phi_N + \phi_0) + T_S \frac{\partial}{\partial \phi_0} (\phi_{SS} - 2\phi_S + \phi_0) + T_E \frac{\partial}{\partial \phi_0} (\phi_{EE} - 2\phi_E + \phi_0) + T_W \frac{\partial}{\partial \phi_0} (\phi_{WW} - 2\phi_W + \phi_0) \right] = 0 \quad (6)
\end{aligned}$$

where

$$\left. \begin{aligned} \beta_i &= E_r h_r \left(\frac{\eta}{Eh} \right)_i & (i = 1, 2, \dots, K) \\ T_i &= E_r h_r \epsilon^2 (\alpha \Delta T \eta)_i & (i = 1, 2, 3, \dots, K) \\ \bar{\beta}_j &= E_r h_r \left(\frac{\eta}{Eh} \right)_j & (j = 1, 2, 3, \dots, M) \end{aligned} \right\} \quad (7)$$

and

E_r reference modulus of elasticity

h_r reference plate thickness

The detailed differentiation with respect to ϕ_0 in equation (6) has not been carried out because the inclusion of certain boundary conditions leads to special results when the general point 0 is related to its immediate neighbors by these boundary conditions. When the value of the solution surface associated with the general point 0 is not related to values associated with other points through the boundary conditions, a general equation holding in the

interior region removed from the boundaries and written about the 0 point is obtained from equation (6) in terms of 13 coefficients as follows:

$$\begin{aligned} & Q_{NN}\phi_{NN} + Q_N\phi_N + Q_{SS}\phi_{SS} + Q_S\phi_S + Q_{EE}\phi_{EE} + Q_E\phi_E + Q_{WW}\phi_{WW} + Q_W\phi_W \\ & + Q_{NW}\phi_{NW} + Q_{NE}\phi_{NE} + Q_{SW}\phi_{SW} + Q_{SE}\phi_{SE} + Q_O\phi_O = G_O \end{aligned} \quad (8)$$

where the coefficients are

$$Q_{NN} = \left(\frac{\epsilon}{\lambda}\right)^4 \beta_N \quad (9)$$

$$Q_{SS} = \left(\frac{\epsilon}{\lambda}\right)^4 \beta_S \quad (10)$$

$$Q_{EE} = \beta_E \quad (11)$$

$$Q_{WW} = \beta_W \quad (12)$$

$$Q_N = -2 \left[\left(\frac{\epsilon}{\lambda}\right)^4 - \mu \left(\frac{\epsilon}{\lambda}\right)^2 \right] (\beta_N + \beta_O) - 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 (\bar{\beta}_{NE/2} + \bar{\beta}_{NW/2}) \quad (13)$$

$$Q_S = -2 \left[\left(\frac{\epsilon}{\lambda}\right)^4 - \mu \left(\frac{\epsilon}{\lambda}\right)^2 \right] (\beta_S + \beta_O) - 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 (\bar{\beta}_{SE/2} + \bar{\beta}_{SW/2}) \quad (14)$$

$$Q_E = -2 \left[1 - \mu \left(\frac{\epsilon}{\lambda}\right)^2 \right] (\beta_E + \beta_O) - 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 (\bar{\beta}_{SE/2} + \bar{\beta}_{NE/2}) \quad (15)$$

$$Q_W = -2 \left[1 - \mu \left(\frac{\epsilon}{\lambda}\right)^2 \right] (\beta_W + \beta_O) - 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 (\bar{\beta}_{SW/2} + \bar{\beta}_{NW/2}) \quad (16)$$

$$Q_{NE} = -\mu \left(\frac{\epsilon}{\lambda}\right)^2 (\beta_E + \beta_N) + 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 \bar{\beta}_{NE/2} \quad (17)$$

$$Q_{NW} = -\mu \left(\frac{\epsilon}{\lambda}\right)^2 (\beta_N + \beta_W) + 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 \bar{\beta}_{NW/2} \quad (18)$$

$$Q_{SE} = -\mu \left(\frac{\epsilon}{\lambda}\right)^2 (\beta_S + \beta_E) + 2(1 + \mu) \left(\frac{\epsilon}{\lambda}\right)^2 \bar{\beta}_{SE/2} \quad (19)$$

$$Q_{SW} = -\mu\left(\frac{\epsilon}{\lambda}\right)^2(\beta_S + \beta_W) + 2(1 + \mu)\left(\frac{\epsilon}{\lambda}\right)^2\bar{\beta}_{SW/2} \quad (20)$$

$$Q_0 = 4\left[\left(\frac{\epsilon}{\lambda}\right)^4 - 2\mu\left(\frac{\epsilon}{\lambda}\right)^2 + 1\right]\beta_0 + \left(\frac{\epsilon}{\lambda}\right)^4(\beta_N + \beta_S) + \beta_E + \beta_W \\ + 2(1 + \mu)\left(\frac{\epsilon}{\lambda}\right)^2(\bar{\beta}_{SW/2} + \bar{\beta}_{SE/2} + \bar{\beta}_{NW/2} + \bar{\beta}_{NE/2}) \quad (21)$$

and

$$G_0 = 2\left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right]T_0 - \left(\frac{\epsilon}{\lambda}\right)^2(T_N + T_S) - T_E - T_W \quad (22)$$

The inclusion of exterior grid points causes the so-called natural boundary conditions (which involve the second and third derivatives of ϕ) to be satisfied as well as is possible for the assumed grid spacing. If there is zero normal force on a boundary, then $\delta(\phi)$ along the boundary is zero so that ϕ may be taken equal to zero along the boundary and the minimization with respect to ϕ does not involve the boundary grid points. For this case equations (8) to (22) apply directly but it should be noted that the integration factor may be zero for some of the coefficients of the off-boundary grid points even though ϕ at the off-boundary grid points is not zero. If there is zero shearing force along the boundary, the boundary condition is equivalent to setting the normal derivative of ϕ along the boundary equal to zero, which means that the off-boundary grid points are determined from the interior points by symmetry relations. These relations must be used in equation (6) before the differentiation is performed. The coefficients to be used in equation (8) for this boundary condition have been derived from equation (6) for several locations of the grid point with respect to the boundary and are shown in appendix A. These values are for a uniform plate but could be extended to the general case by the use of the same symmetry conditions in conjunction with equation (6). Also included in appendix A are the coefficients for a general interior point.

Equation (8), when properly written to include the effects of the imposed boundary conditions, defines a set of R algebraic equations in R unknowns which may be written in matrix form as

$$[Q]\{\phi\} = \{G\} \quad (23)$$

The in-plane forces are then determined by solution of equation (23) for ϕ_j and direct substitution of the results into equations (2) and (4).

NUMERICAL EVALUATION OF METHOD

The applicability and accuracy of the finite-difference method is evaluated by applying it to determine the midplane force distribution for a thin, flat plate with a "tent" type of temperature distribution as shown in figure 3. This problem was chosen because an approximate solution and experimental results are presented in reference 1 to provide a comparison of the results.

Two grid spacings are used to provide some indication of the convergence. The grid spacing shown in figure 3 is referred to as fine grid spacing and results in a solution surface which can be determined at grid stations 1 to 24. A grid spacing having intervals which are twice the size of the fine grid intervals is referred to as coarse grid spacing and results in a solution surface which can be determined at grid stations 1, 3, 5, 13, 15, and 17.

The actual plate occupies the region $-a \leq x \leq a$, $-b \leq y \leq b$; but, because of plate and temperature symmetry, it is sufficient to consider only one quadrant. The quadrant considered is located in the region $0 \leq x \leq a$, $0 \leq y \leq b$ where the temperature distribution is represented mathematically by

$$\Delta T = \Delta T_1 \left(1 - \frac{y}{b} \right)$$

The plate is assumed to be free of stress along the boundaries $x = a$ and $y = b$ and, because of symmetry, $N_{xy} = 0$ along $x = 0$ and $y = 0$. Thus, the boundary conditions are

$$\phi = \frac{\partial \phi}{\partial x} = 0 \quad (x = a)$$

$$\phi = \frac{\partial \phi}{\partial y} = 0 \quad (y = b)$$

$$\frac{\partial \phi}{\partial x} = 0 \quad (x = 0)$$

$$\frac{\partial \phi}{\partial y} = 0 \quad (y = 0)$$

By considering these boundary conditions, the set of equations for the determination of ϕ_i may be written directly from appendix A. These equations for the coarse and fine grid spacing are presented in appendix B, respectively.

RESULTS AND DISCUSSION

Pictorial representations of the results are shown in figures 4, 5, 6, and 7. Figure 4 shows the solution surface defined by the Airy stress function;

figure 5 shows the distribution of N_x ; figure 6 shows the distribution of N_y ; and figure 7 shows the distribution of N_{xy} . Note that the point of view chosen in each figure is that which will best exhibit the characteristics of the variables.

The results of the finite-difference solutions and the approximate solution of reference 1 are compared in figures 8, 9, 10, and 11. Figure 8 shows the variation of the midplane force N_x with y/b at $x/a = 0$. Figure 9 shows the variation of the midplane force N_y with x/a at $y/b = 0$. Figure 10 shows the variation of N_x with x/a for $y/b = 0$ and $y/b = 1$. Figure 11 shows the variation of N_x with y/b at $x/a = 0.67$ together with experimental data for $x/a = 0.7$. Comparison of the results for the coarse and fine grid networks indicates that the finite-difference calculations are converging rapidly as the number of grid stations is increased. It is noted that the stresses calculated from the finite-difference solutions are generally in good agreement with the results of the approximate solution of reference 1. However, the trend of the two finite-difference solutions and the comparison with experiment shown in figure 11 suggest that the finite-difference results are more accurate than the approximate results of reference 1.

CONCLUDING REMARKS

The applicability of a variational technique to obtain directly the finite-difference equations for the Airy stress function associated with thermally induced midplane forces has been evaluated. The evaluation was made by applying the method to determine the stress distribution for a plate configuration and temperature distribution for which an approximate solution and experimental data are available. Good agreement with these results was found. The results of calculations based on a rather coarse and a fine grid network indicated that the finite-difference solutions converge rapidly as the number of grid stations is increased.

It is concluded that the formulation of the finite-difference equations from the complementary strain energy, with its inherent advantage that only the "geometric" boundary conditions need be imposed on the stress function at the boundary, is an excellent method for the calculation of thermally induced midplane forces. The resulting generic difference equation for interior rectangular grid points of plates with variable properties is recorded for convenience. The treatment of points near plate boundaries has been presented in convenient form for the important case of a rectangular plate of constant thickness.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 25, 1964.

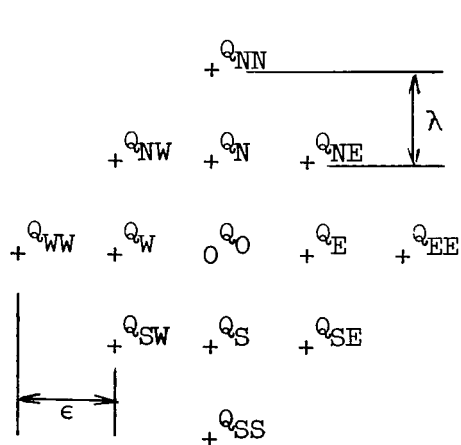
APPENDIX A

VALUES OF COEFFICIENTS TO BE USED IN EQUATION (8)

FOR A BOUNDARY HAVING ZERO SHEARING FORCE

The coefficients for a general interior point are given first. For subsequent cases only coefficients that have changed from those for the general case are given. The boundary condition has been used to eliminate the values of ϕ for the exterior points; hence the ϕ_1 at all points exterior to the plate now have zero coefficients in equation (8).

For the general interior point:



$$Q_{NN} = Q_{SS} = \left(\frac{\epsilon}{\lambda}\right)^4$$

$$Q_{EE} = Q_{WW} = 1$$

$$Q_{NE} = Q_{NW} = Q_{SE} = Q_{SW} = 2\left(\frac{\epsilon}{\lambda}\right)^2$$

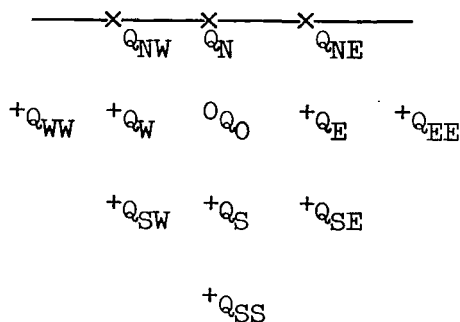
$$Q_N = Q_S = -4\left(\frac{\epsilon}{\lambda}\right)^2 \left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right]$$

$$Q_E = Q_W = -4 \left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right]$$

$$Q_O = 6 + 8\left(\frac{\epsilon}{\lambda}\right)^2 + 6\left(\frac{\epsilon}{\lambda}\right)^4$$

$$G_O = 2 \left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right] T_O - \left(\frac{\epsilon}{\lambda}\right)^2 (T_N + T_S) - T_E - T_W$$

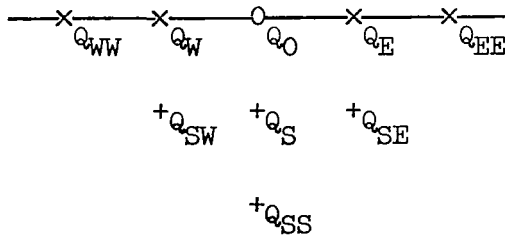
For a point 1 grid space from the boundary:



$$Q_O = 6 + 8\left(\frac{\epsilon}{\lambda}\right)^2 + 7\left(\frac{\epsilon}{\lambda}\right)^4$$

$$G_O = 2 \left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right] T_O - \left(\frac{\epsilon}{\lambda}\right)^2 (2T_N + T_S) - T_E - T_W$$

For a point on the boundary:

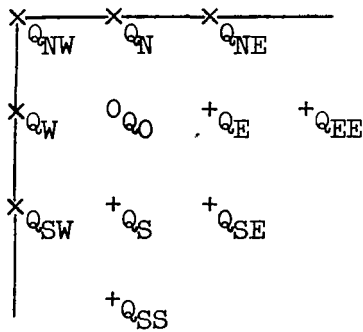


$$Q_{EE} = Q_{WW} = \frac{1}{2}$$

$$Q_E = Q_W = -2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right]$$

$$Q_O = 3 + 4 \left(\frac{\epsilon}{\lambda} \right)^2 + 3 \left(\frac{\epsilon}{\lambda} \right)^4$$

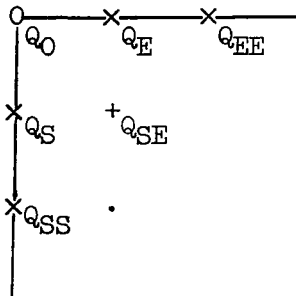
For a point 1 grid space from the corner:



$$Q_O = 7 + 8 \left(\frac{\epsilon}{\lambda} \right)^2 + 7 \left(\frac{\epsilon}{\lambda} \right)^4$$

$$G_O = 2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right] T_O - \left(\frac{\epsilon}{\lambda} \right)^2 (2T_N + T_S) - T_E - 2T_W$$

For a point on the corner:



$$Q_{EE} = \frac{1}{2}$$

$$Q_E = -2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right]$$

$$Q_{SS} = \frac{1}{2} \left(\frac{\epsilon}{\lambda} \right)^4$$

$$Q_S = -2 \left(\frac{\epsilon}{\lambda} \right)^2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right]$$

$$Q_O = \frac{3}{2} + 2 \left(\frac{\epsilon}{\lambda} \right)^2 + \frac{3}{2} \left(\frac{\epsilon}{\lambda} \right)^4$$

For a point $1\frac{1}{2}$ grid spaces from the boundary:

$$\begin{array}{ccccc}
 & +Q_{NW} & +Q_N & +Q_{NE} & \\
 +Q_{WW} & +Q_W & OQ_O & +Q_E & +Q_{EE} \\
 & +Q_{SW} & +Q_S & +Q_{SE} & \\
 & & +Q_{SS} & &
 \end{array}$$

$$Q_N = -5\left(\frac{\epsilon}{\lambda}\right)^4 - 2\left(\frac{\epsilon}{\lambda}\right)^2$$

For a point $1\frac{1}{2}$ grid spaces from the corner:

$$\begin{array}{cccc}
 +Q_{NW} & +Q_N & +Q_{NE} & \\
 +Q_W & OQ_O & +Q_E & +Q_{EE} \\
 +Q_{SW} & +Q_S & +Q_{SE} & \\
 & +Q_{SS} & &
 \end{array}$$

$$Q_N = -5\left(\frac{\epsilon}{\lambda}\right)^4 - 2\left(\frac{\epsilon}{\lambda}\right)^2$$

$$Q_W = -3 - 4\left(\frac{\epsilon}{\lambda}\right)^2$$

For a point $\frac{1}{2}$ grid space from the corner:

$$\begin{array}{ccc}
 OQ_O & +Q_E & +Q_{EE} \\
 +Q_S & +Q_{SE} & \\
 +Q_{SS} & &
 \end{array}$$

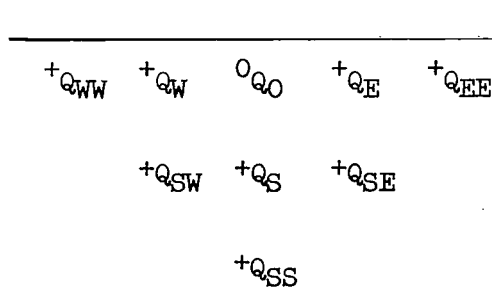
$$Q_E = -3 - 2\left(\frac{\epsilon}{\lambda}\right)^2$$

$$Q_S = -3\left(\frac{\epsilon}{\lambda}\right)^4 - 2\left(\frac{\epsilon}{\lambda}\right)^2$$

$$Q_O = 2\left[1 + \left(\frac{\epsilon}{\lambda}\right)^2 + \left(\frac{\epsilon}{\lambda}\right)^4\right]$$

$$G_O = \left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right]T_O - \left(\frac{\epsilon}{\lambda}\right)^2(T_S - T_N) - T_E + T_W$$

For a point $\frac{1}{2}$ grid space from the boundary:



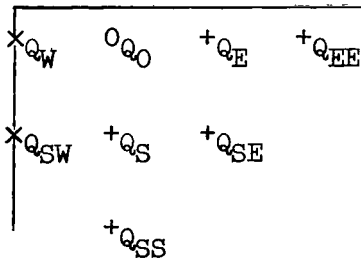
$$Q_E = Q_W = -4 - 2\left(\frac{\epsilon}{\lambda}\right)^2$$

$$Q_S = -4\left(\frac{\epsilon}{\lambda}\right)^2 - 3\left(\frac{\epsilon}{\lambda}\right)^4$$

$$Q_O = 6 + 4\left(\frac{\epsilon}{\lambda}\right)^2 + 2\left(\frac{\epsilon}{\lambda}\right)^4$$

$$G_O = \left[2 + \left(\frac{\epsilon}{\lambda}\right)^2\right]T_O - \left(\frac{\epsilon}{\lambda}\right)^2(T_S - T_N) - T_E - T_W$$

For a point 1 grid space from one boundary and $\frac{1}{2}$ grid space from another boundary:



$$Q_E = Q_W = -4 - 2\left(\frac{\epsilon}{\lambda}\right)^2$$

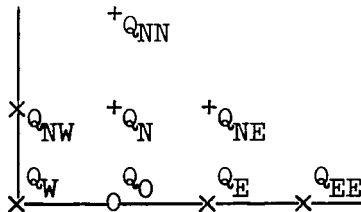
$$Q_S = -4\left(\frac{\epsilon}{\lambda}\right)^2 - 3\left(\frac{\epsilon}{\lambda}\right)^4$$

$$Q_O = 7 + 2\left(\frac{\epsilon}{\lambda}\right)^4 + 4\left(\frac{\epsilon}{\lambda}\right)^2$$

$$G_O = \left[2 + \left(\frac{\epsilon}{\lambda}\right)^2\right]T_O - \left(\frac{\epsilon}{\lambda}\right)^2(T_S - T_N) - T_E - 2T_W$$

For a point on one boundary and 1 grid space from another boundary:

Case 1:



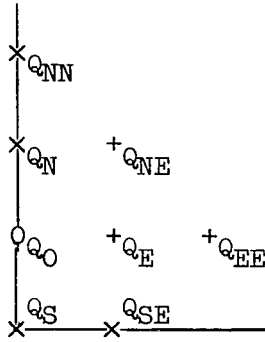
$$Q_{EE} = \frac{1}{2}$$

$$Q_E = Q_W = -2\left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right]$$

$$Q_O = 3\frac{1}{2} + 3\left(\frac{\epsilon}{\lambda}\right)^4 + 4\left(\frac{\epsilon}{\lambda}\right)^2$$

$$G_O = 2\left[1 + \left(\frac{\epsilon}{\lambda}\right)^2\right]T_O - \left(\frac{\epsilon}{\lambda}\right)^2(T_N + T_S) - T_E - 2T_W$$

Case 2:



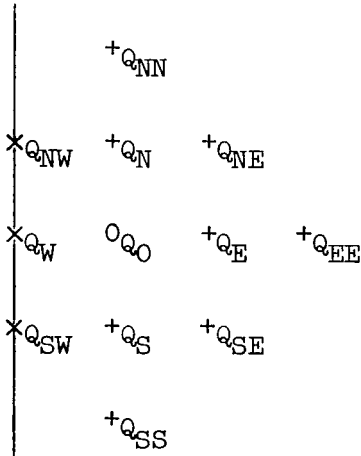
$$Q_{NN} = \frac{1}{2} \left(\frac{\epsilon}{\lambda} \right)^4$$

$$Q_N = Q_S = -2 \left(\frac{\epsilon}{\lambda} \right)^2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right]$$

$$Q_O = 3 + 3 \frac{1}{2} \left(\frac{\epsilon}{\lambda} \right)^4 + 4 \left(\frac{\epsilon}{\lambda} \right)^2$$

$$G_O = 2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right] T_O - \left(\frac{\epsilon}{\lambda} \right)^2 (T_N + 2T_S) - T_E - T_W$$

For a point 1 grid space from the boundary:



$$Q_O = 7 + 8 \left(\frac{\epsilon}{\lambda} \right)^2 + 6 \left(\frac{\epsilon}{\lambda} \right)^4$$

$$G_O = 2 \left[1 + \left(\frac{\epsilon}{\lambda} \right)^2 \right] T_O - \left(\frac{\epsilon}{\lambda} \right)^2 (T_N + T_S) - T_E - 2T_W$$

APPENDIX B

FINITE-DIFFERENCE EQUATIONS FOR STRESS FUNCTION

Finite-difference equations for determining ϕ_i can be written for the rectangular plate shown in figure 3.

The equation for the coarse grid system is

$$\begin{bmatrix} 5 & -4 & .5 & -4 & 2 & 0 \\ -4 & 10.5 & -4 & 2 & -8 & 2 \\ .5 & -4 & 10.5 & 0 & 2 & -8 \\ -4 & 2 & 0 & 11.0 & -8 & 1 \\ 2 & -8 & 2 & -8 & 23 & -8 \\ 0 & 2 & -8 & 1 & -8 & 23 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_3 \\ \phi_5 \\ \phi_{13} \\ \phi_{15} \\ \phi_{17} \end{Bmatrix} = \frac{\alpha E_T h_T \Delta T_1 a^2}{9} \begin{Bmatrix} .25 \\ .5 \\ .5 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The equation for the fine grid system is

5	-4	.5	0	0	0	-4	2	0	0	0	0	.5	0	0	0	0	0	0	0	0	0	0	ϕ_1	.125
10.5	-4	.5	0	0	0	2	-8	2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	ϕ_2	.25
	10	-4	.5	0	0	2	-8	2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	ϕ_3	.25
		10	-4	.5	0	0	2	-8	2	0	0	0	0	1	0	0	0	0	0	0	0	0	ϕ_4	.25
			10	-4	0	0	0	2	-8	2	0	0	0	0	1	0	0	0	0	0	0	0	ϕ_5	.25
				10.5	0	0	0	0	2	-8	0	0	0	0	0	1	0	0	0	0	0	0	ϕ_6	.25
					10.5	-8	1	0	0	0	-4	2	0	0	0	0	.5	0	0	0	0	0	ϕ_7	0
						22	-8	1	0	0	2	-8	2	0	0	0	0	1	0	0	0	0	ϕ_8	0
							21	-8	1	0	0	2	-8	2	0	0	0	0	1	0	0	0	ϕ_9	0
								21	-8	1	0	0	2	-8	2	0	0	0	0	1	0	0	ϕ_{10}	0
									21	-8	0	0	0	2	-8	2	0	0	0	0	1	0	ϕ_{11}	0
										22	0	0	0	0	2	-8	0	0	0	0	0	1	ϕ_{12}	$= \frac{\alpha E_r h_r \Delta T_1 a^2}{36}$
											10	-8	1	0	0	0	-4	2	0	0	0	0	ϕ_{13}	0
												21	-8	1	0	0	2	-8	2	0	0	0	ϕ_{14}	0
													20	-8	1	0	0	2	-8	2	0	0	ϕ_{15}	0
														20	-8	1	0	0	2	-8	2	0	ϕ_{16}	0
															20	-8	0	0	0	2	-8	2	ϕ_{17}	0
																21	0	0	0	0	2	-8	ϕ_{18}	0
																	10.5	-8	1	0	0	2	ϕ_{19}	0
																		22	-8	1	0	0	ϕ_{20}	0
																			21	-8	1	0	ϕ_{21}	0
																				21	-8	1	ϕ_{22}	0
																					21	-8	ϕ_{23}	0
																						22	ϕ_{24}	0

Symmetric

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6. Walton, William C., Jr.: Applications of a General Finite-Difference Method for Calculating Bending Deformations of Solid Plates. NASA TN D-536, 1960.
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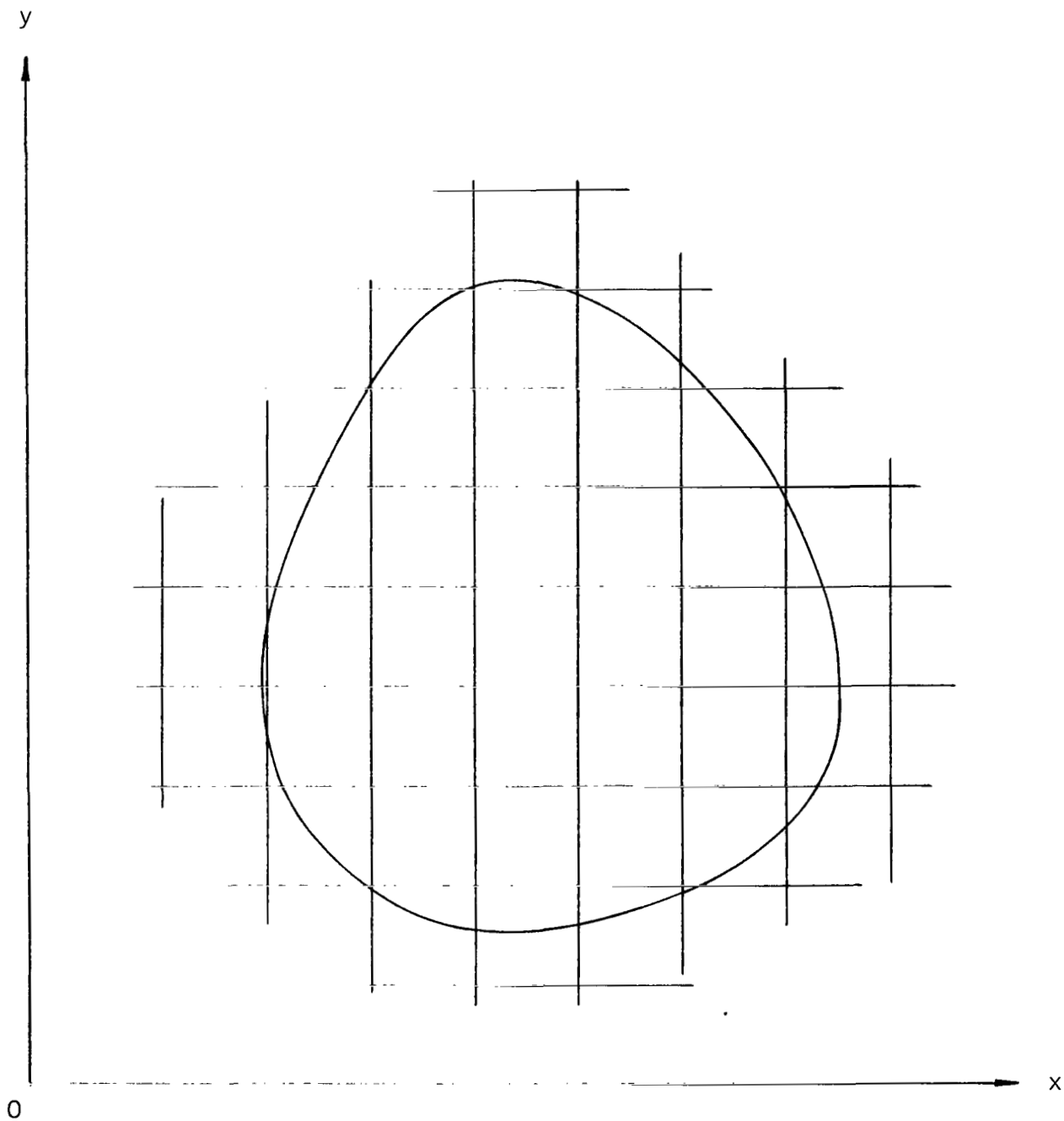


Figure 1.- General planform of plate showing possible grid network.

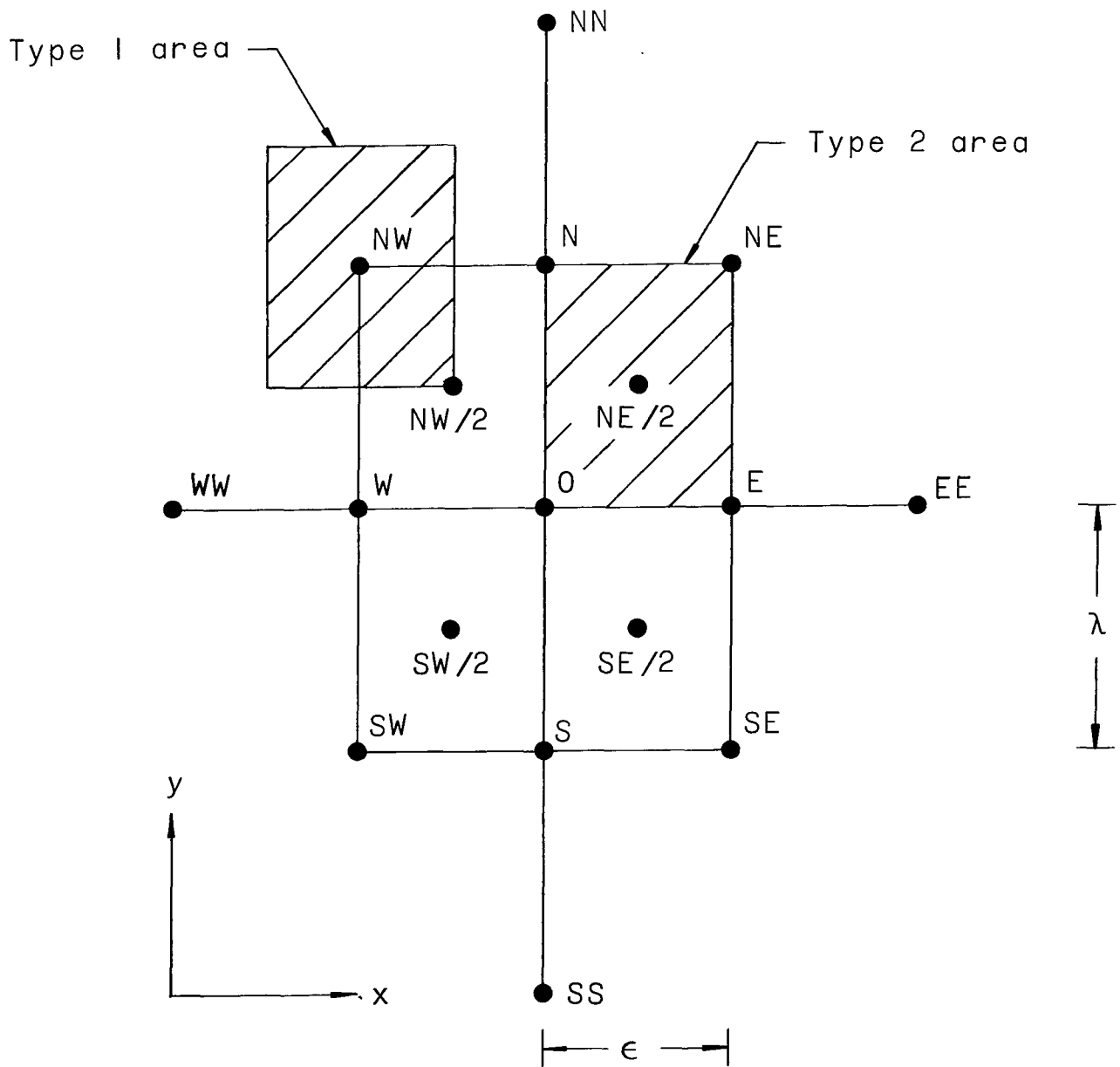
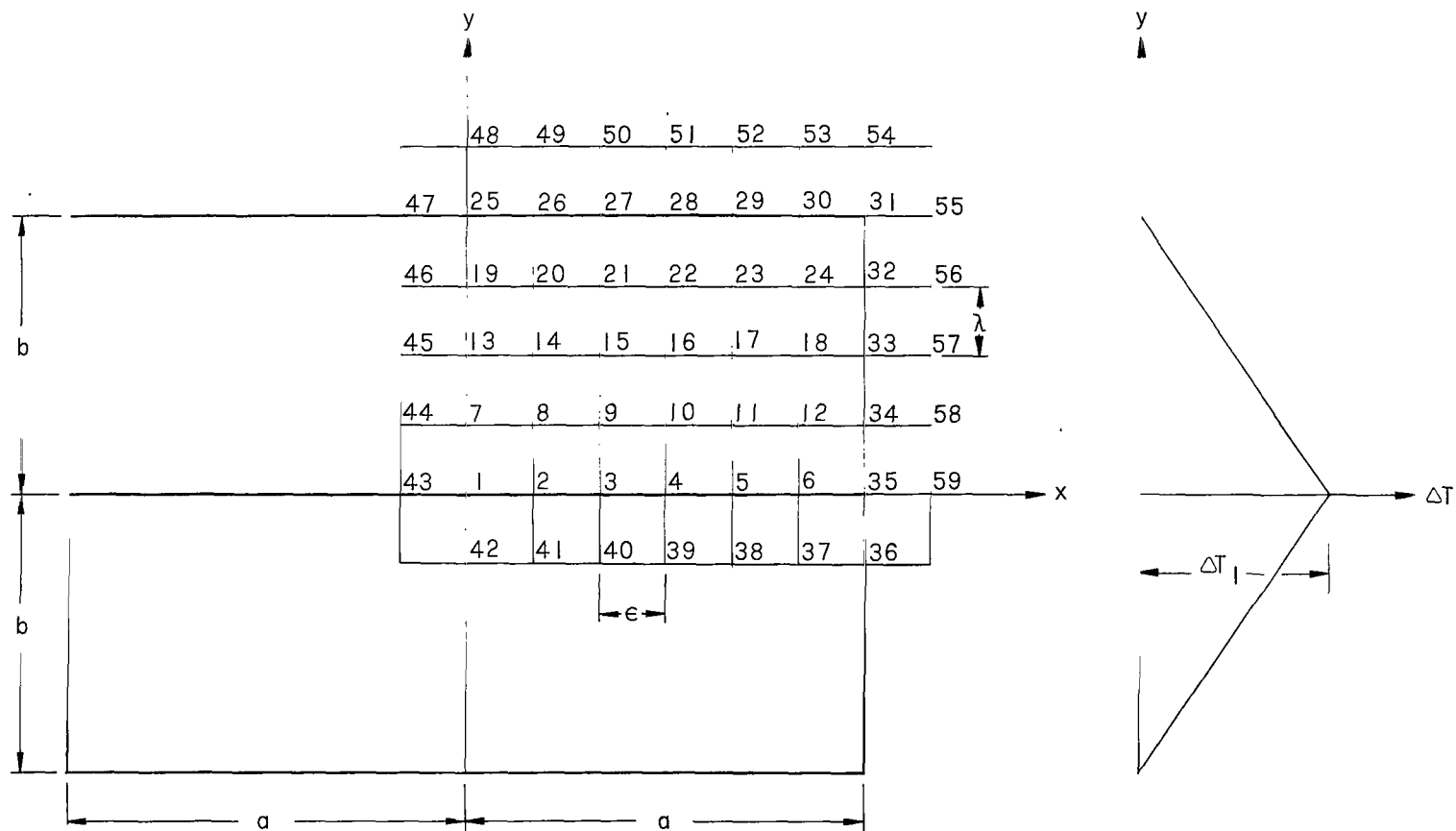


Figure 2.- General point O , its neighbors, and illustrations of type 1 and type 2 integration areas.



(a) Plate planform and grid network.
 $a/b = 3/2$; $\epsilon = \lambda = a/6$.

(b) Temperature distribution.
 $\Delta T = \Delta T_1(1 - y/b)$;
 $0 \leq y \leq b$; $0 \leq x \leq a$.

Figure 3.- Grid spacing and temperature distribution used in evaluating the finite-difference technique.

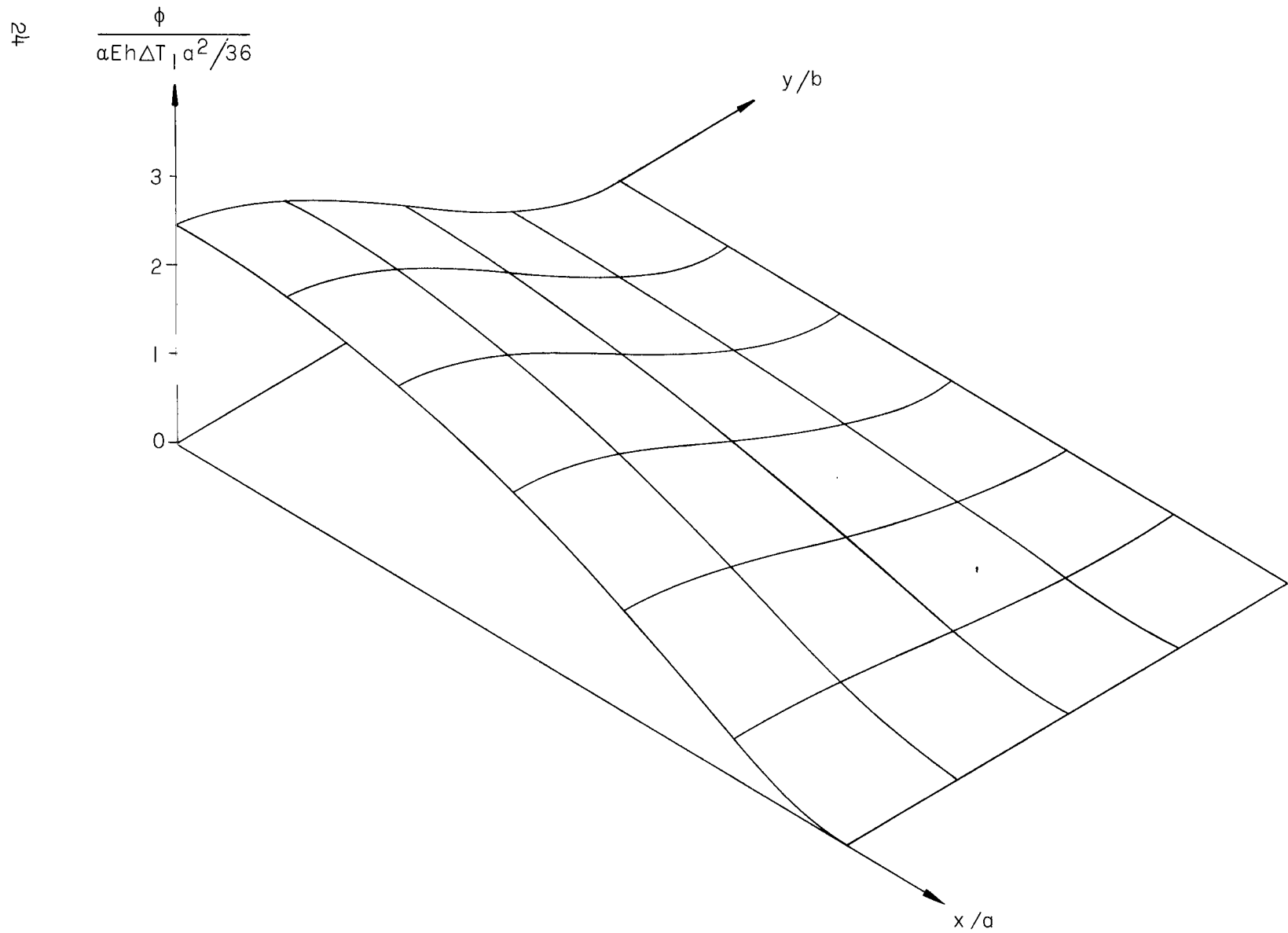


Figure 4.- Solution surface defined by Airy stress function for rectangular plate with $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $x/a = 0$ and $y/b = 0$. $a/b = 3/2$; $\mu = 1/3$; $\epsilon = \lambda = a/6$.

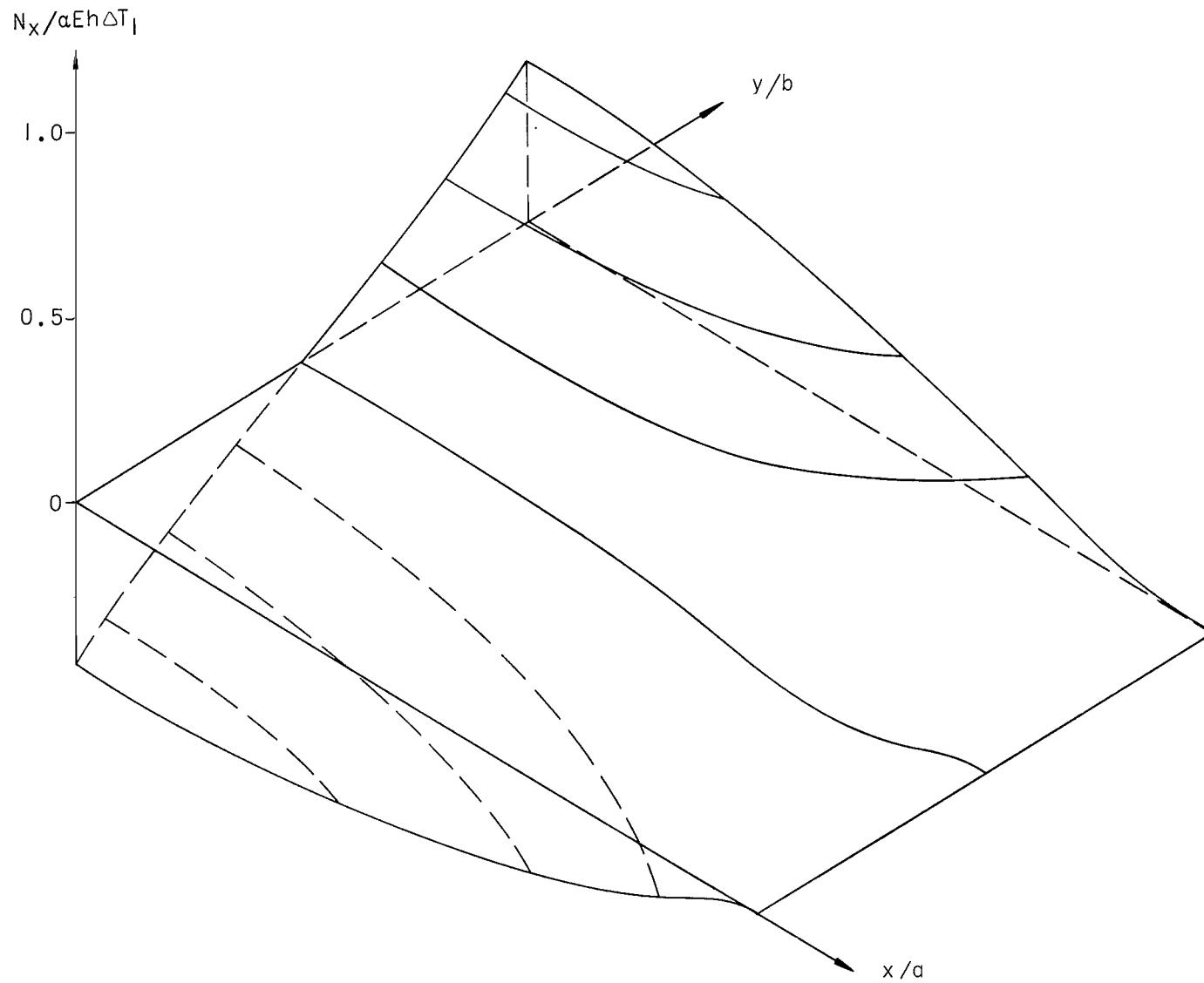


Figure 5.- Distribution of midplane force intensity N_x due to temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $x/a = 0$ and $y/b = 0$.

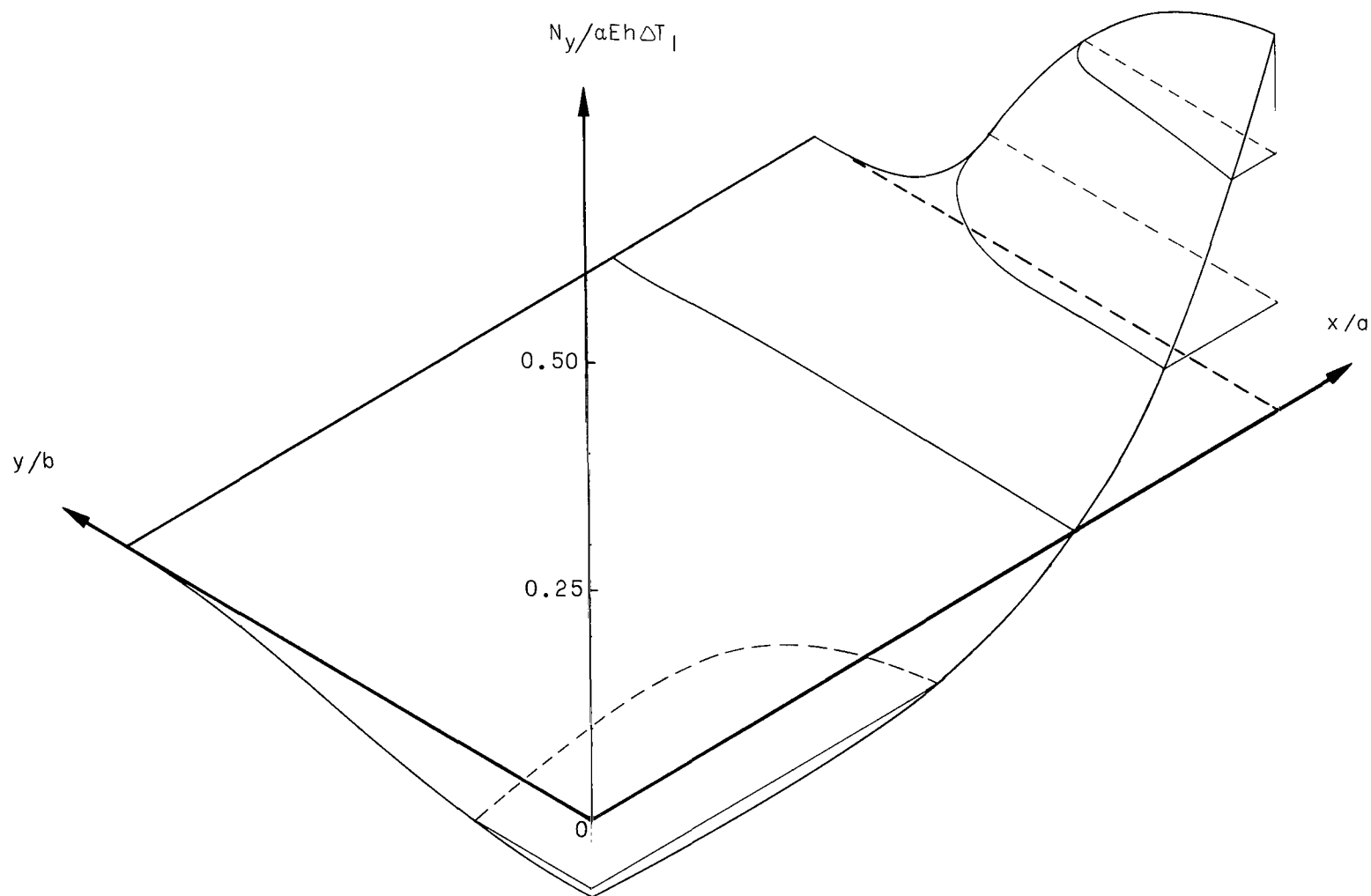


Figure 6.- Distribution of midplane force intensity N_y due to temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $x/a = 0$ and $y/b = 0$.

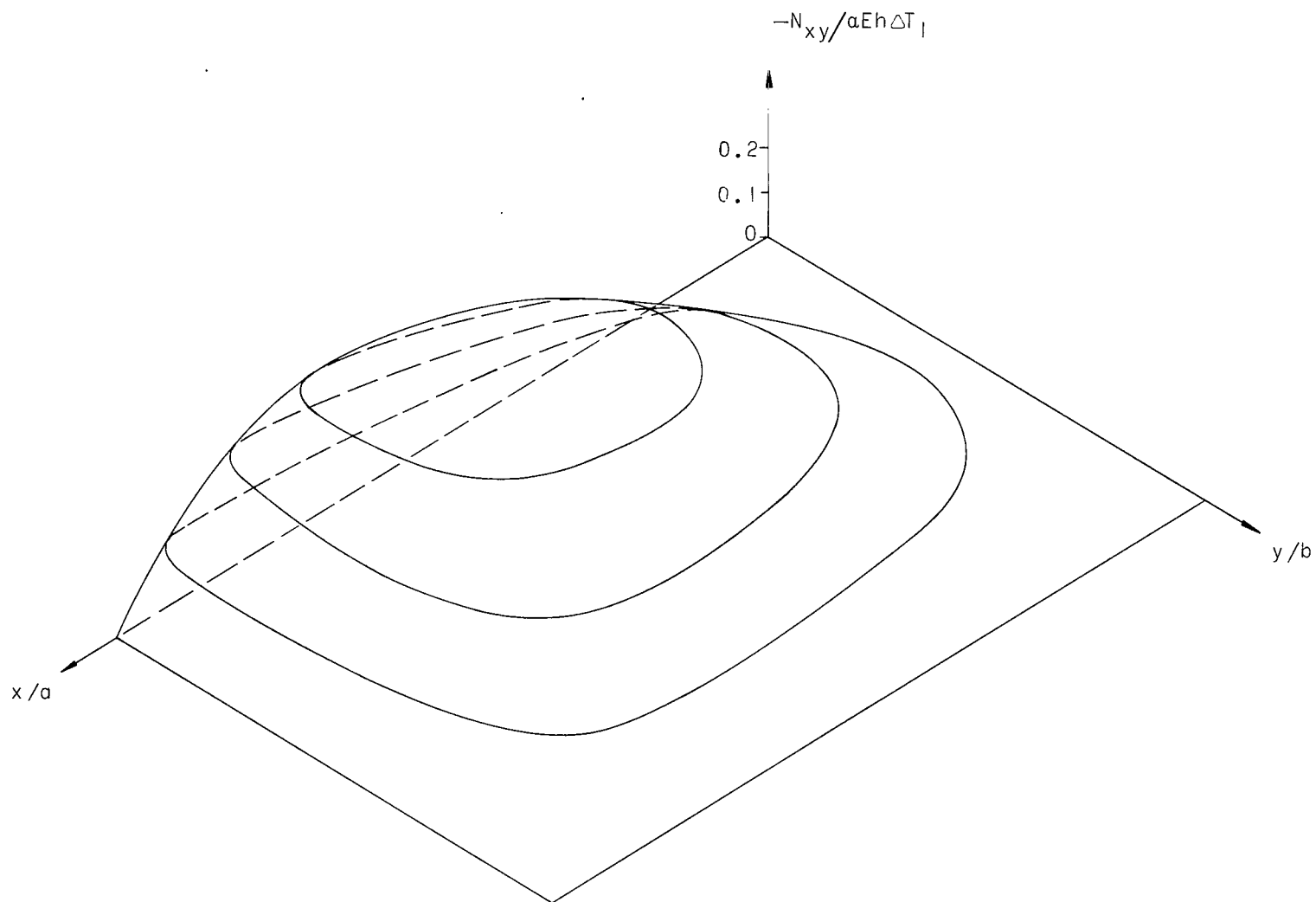


Figure 7.- Distribution of midplane shear force intensity N_{xy} due to temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Anti-symmetric about $x/a = 0$ and $y/b = 0$.

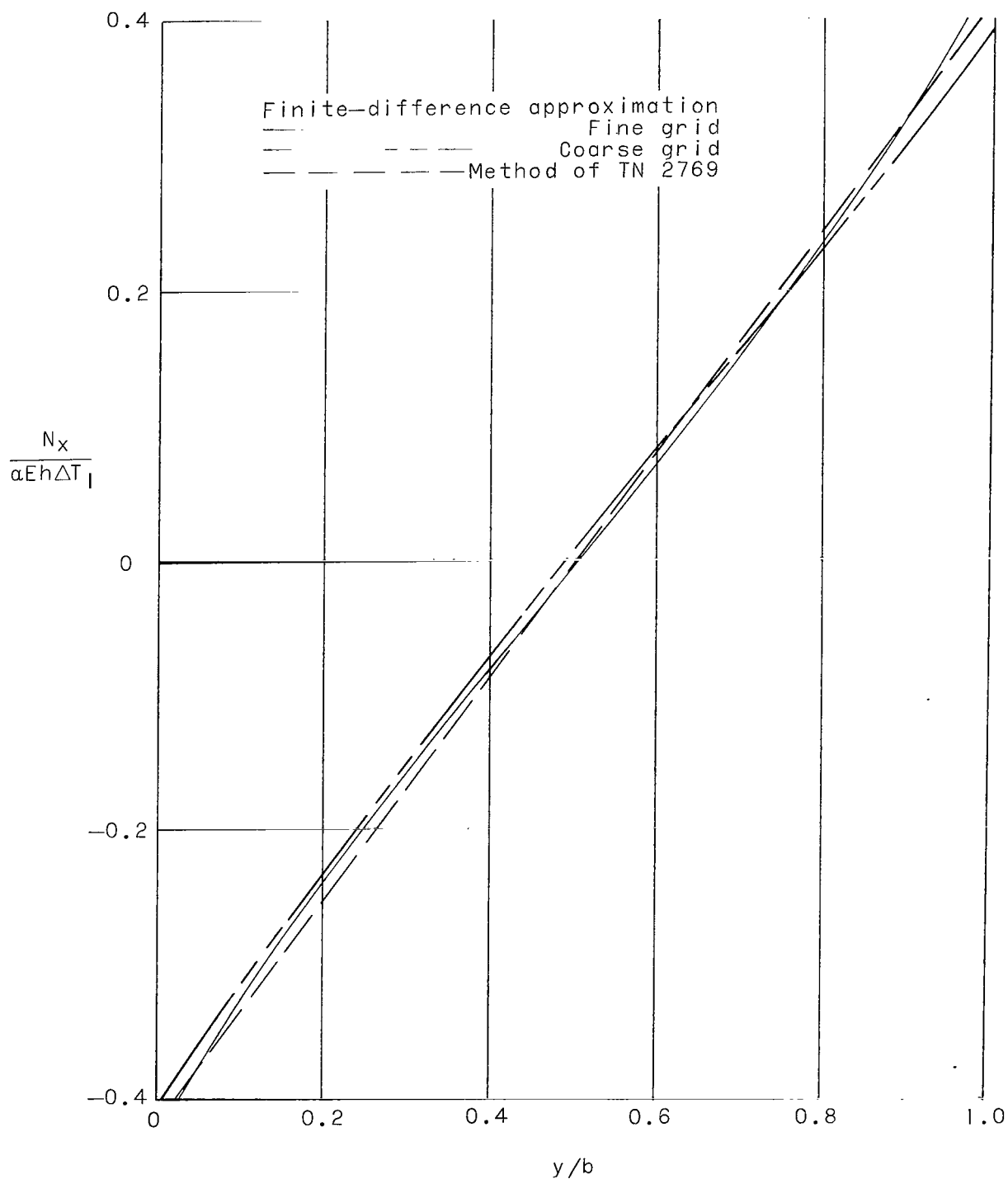


Figure 8.- Comparison of variation of midplane force intensity N_x with y/b at $x/a = 0$ for temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $y/b = 0$.

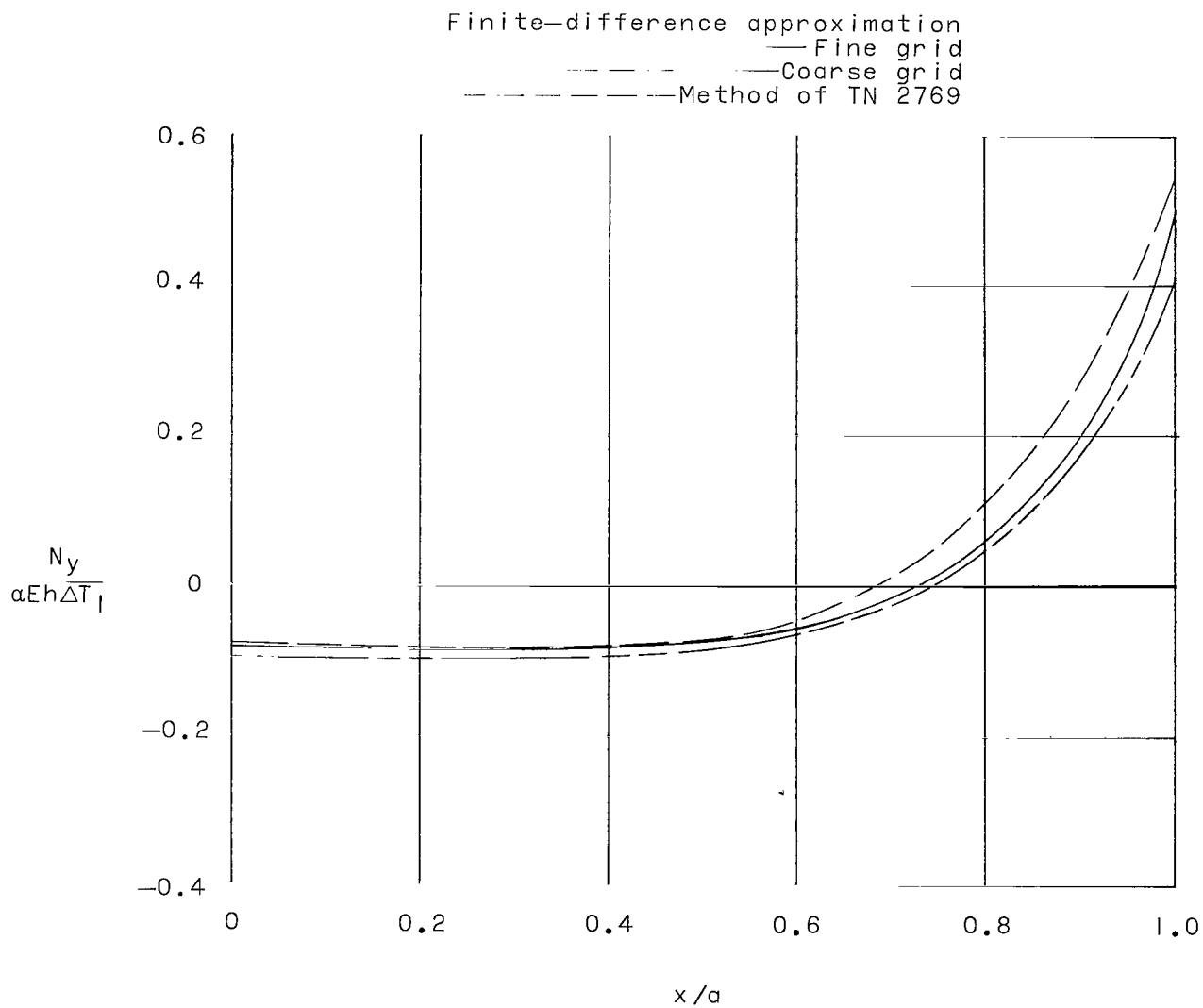


Figure 9.- Comparison of variation of midplane force intensity N_y with x/a at $y/b = 0$ for temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $x/a = 0$.

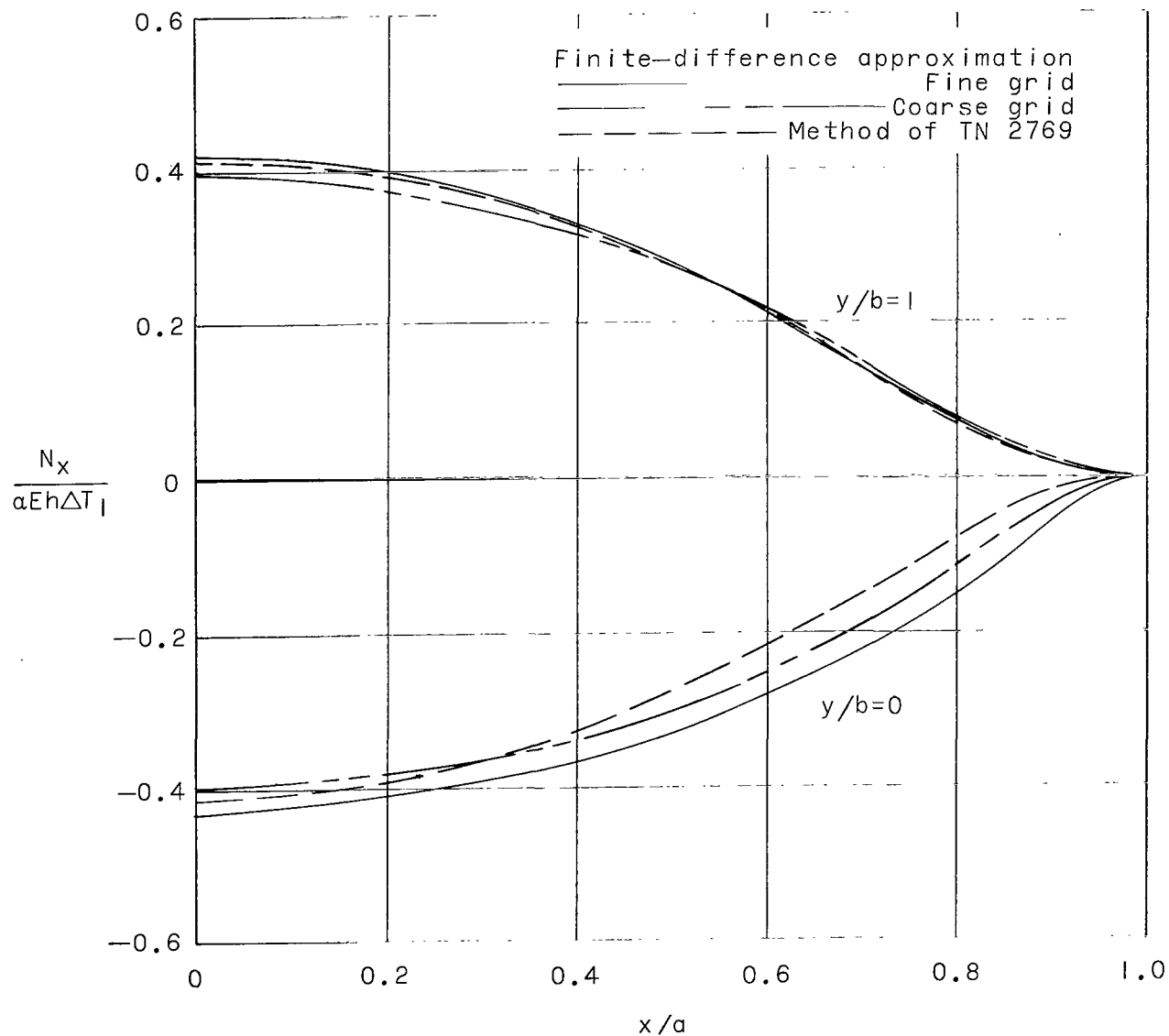


Figure 10.- Comparison of variation of midplane force intensity N_x with x/a at $y/b = 0$ and $y/b = 1$ for temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $x/a = 0$.

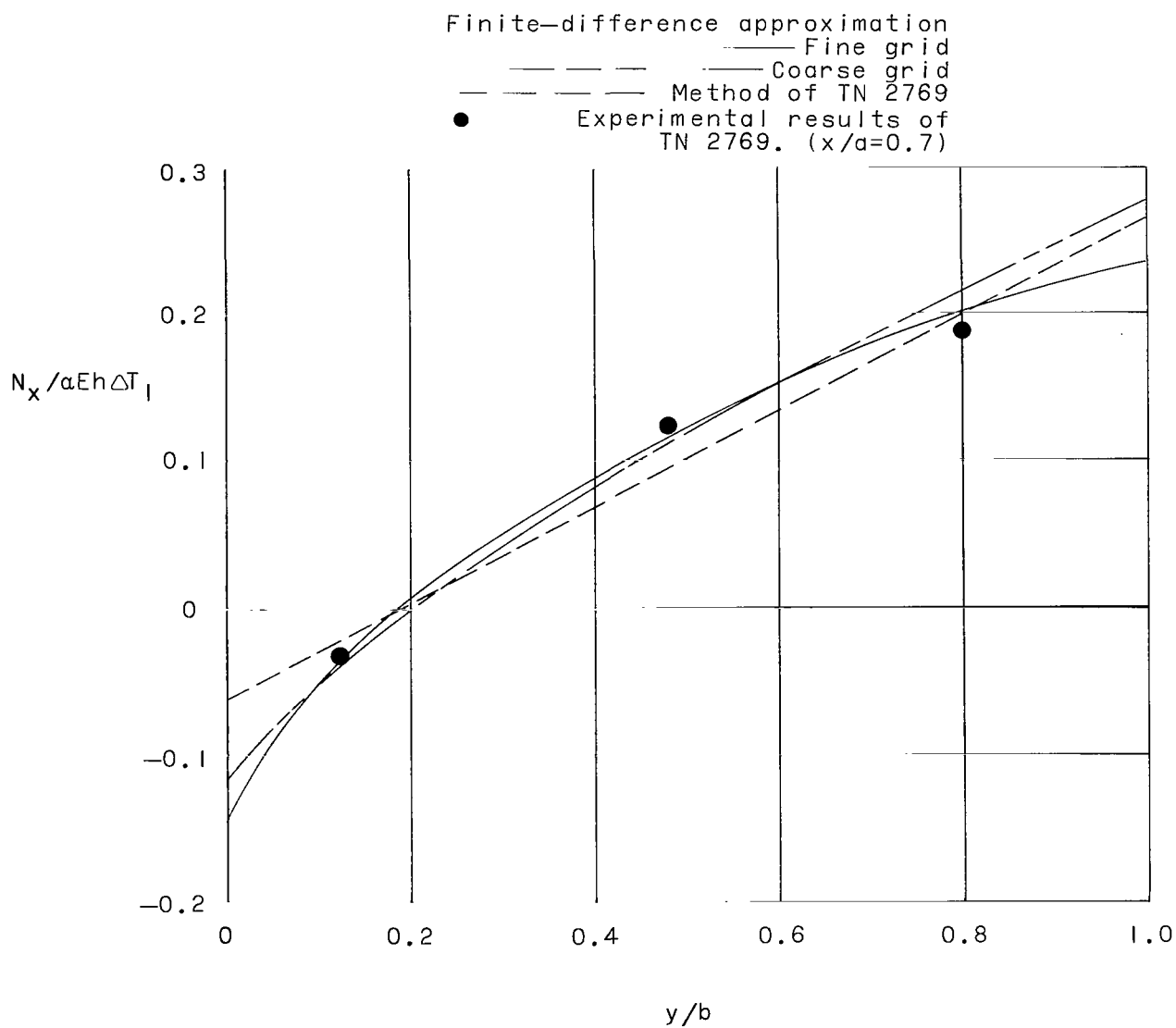


Figure 11.- Comparison of variation of midplane force intensity N_x with y/b at $x/a = 0.67$ for temperature distribution $\Delta T = \Delta T_1(1 - y/b)$. Symmetric about $y/b = 0$.

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